Μερικές Διαφορικές Εξισώσεις Α΄ Τάξης

- Να λυθούν οι παρακάτω μερικές διαφορικές εξισώσεις

i) $2x\, Z_x + 3y\cdot Z_y = \log x$, $x>0$ και $y>0$

Με τη βοήθεια του μετασχηματισμού

$\xi = \log x$ και $\eta = \log y$

ii) $Z_x - Z_y = 1$, $(x,y)\in\mathbb{R}^2$ που πληρεί την συνθήκη

$Z(x,0) = \sin x$.

ΛΥΣΗ

i)
$$\left.\begin{aligned} \xi &= \log x \\ \eta &= \log y \end{aligned}\right| \quad \begin{aligned} Z_x &= \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = Z_\xi\cdot\frac{1}{x} \\ Z_y &= \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = Z_\eta\cdot\frac{1}{y} \end{aligned}$$

Αντικαθιστώντας, παίρνουμε:

$$2x\cdot\frac{Z_\xi}{x} + 3y\,\frac{Z_\eta}{y} = \overset{\xi}{\log x} \Rightarrow 2Z_\xi + 3\cdot Z_\eta = \xi \quad , (\xi,\eta)\in\mathbb{R}^2 \ (E_1)$$

οι λύσεις της δε, δίνονται από τη σχέση (1)

$$\frac{d\eta}{d\xi} = \frac{3}{2} \Rightarrow 2d\eta - 3d\xi = 0 \Rightarrow 2\eta - 3\xi = c \quad (1)$$

Έτσι, θέτουμε $u = \xi$ και $w = 2\eta - 3\xi$

$$Z_\xi = \frac{\partial z}{\partial \xi} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial \xi} + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial \xi} = Z_u + Z_w\cdot(-3) = Z_u - 3\cdot Z_w.$$

$$Z_\eta = \frac{\partial z}{\partial \eta} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial \eta} + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial \eta} = \cancel{Z_u\cdot 0}^{\,0} + Z_w\cdot 2 = 2\cdot Z_w$$

Αντικαθιστώντας στην (E_1) έχουμε:

$$2\cdot(Z_u - 3\cdot Z_w) + 3\cdot 2\cdot Z_w = u \Rightarrow 2Z_u = u \Rightarrow \frac{\partial z}{\partial u} = \frac{u}{2} \Rightarrow$$

$$\Rightarrow \quad z(u,w) = \frac{u^2}{4} + z(0,w) \Rightarrow$$

$$\Rightarrow \quad z(\xi,\eta) = \frac{\xi^2}{4} + z(0, 2\eta - 3\xi) \Rightarrow$$

$$\Rightarrow \quad z(x,y) = \frac{\log^2 x}{4} + z(0, 2\log y - 3\log x)$$

ii) $z_x - z_y = 1$

Οι λύσεις της Δ.Ε

$$\frac{dx}{dy} = \frac{1}{-1} \Rightarrow dx = -dy \Rightarrow dy = -dx \Rightarrow$$

$$\Rightarrow \int dy = -\int dx \Rightarrow y = -x + d \Rightarrow d = y + x$$

Θέτω, $\eta = y$, $\xi = x+y$ $\quad\Big\|\quad \left.\begin{aligned} z_x &= z_\xi + 0\cdot z_\eta = z_\xi \\ z_y &= z_\xi + z_\eta = z_\xi + z_\eta \end{aligned}\right\}$ Αντικαθιστώντας →

$$\rightarrow z_\xi - \cancel{z_\xi} - z_\eta = 1 \Rightarrow z_\eta = -1 \Rightarrow z(\xi,\eta) = -\eta + z(\xi,0) \Rightarrow$$

$$\Rightarrow z(x,y) = -y + z(x+y,0)$$

$$z(x,0) = 0 + z(x,0) = \sin x \Rightarrow z(x+y,0) = \sin(x+y)$$

Άρα, $z(x,y) = y + \sin(x+y)$.

$x z_x + (x+y) z_y = 1, \quad x>0, \; 0<y<1, \quad z(1,y) = y$

ΛΥΣΗ

Οι λύσεις της δ.ε

$\frac{dy}{dx} = \frac{x+y}{x} \Rightarrow y' = \frac{x+y}{x}$ ομογενής

θέτω $y = z_1 x \Rightarrow y' = z_1' x + z_1$

Αντικαθιστούμε:

$z_1' x + z_1 = \frac{x + z_1 x}{x} \Longrightarrow z_1' x + z_1 = \frac{1+z_1}{1} \Rightarrow z_1' x = 1 \Rightarrow$

$\Rightarrow z_1' = \frac{1}{x} \Rightarrow z_1 = \int_1^x \frac{1}{t} dt \Rightarrow z_1 = \log x \quad \forall x>0$

Άρα, $y = x \cdot \log x$

Οι λύσεις θα δίνονται από τη σχέση:

$x \cdot \log x - y = d.$

$\xi = x$

$n = x\log x - y$

$z_x = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial n} \cdot \frac{\partial n}{\partial x} = z_\xi + z_n \cdot (\log x + 1)$

$z_y = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial n} \cdot \frac{\partial n}{\partial y} = \cancel{z_\xi \cdot 0}^{0} + z_n \cdot (-1) = -z_n$

$x \cdot (z_\xi + z_n(1+\log x)) + (x + x\log x) \cdot (-z_n) = 1 \Rightarrow$

$\Rightarrow x z_\xi + \cancel{x z_n} + \cancel{x z_n \log x} - \cancel{x z_n} - \cancel{x z_n \cdot \log x} = 1 \Rightarrow$

$\Rightarrow x z_\xi = 1 \Rightarrow z_\xi = \frac{1}{x} \Rightarrow \frac{\partial}{\partial \xi} \cdot z(\xi, n) = \frac{1}{x} = \frac{1}{\xi} \Rightarrow$

$\Rightarrow \frac{\partial}{\partial \xi} z(\xi, n) = \frac{1}{\xi} \Rightarrow z(\xi, n) = \int \frac{1}{\xi} d\xi + z(0, n) \Rightarrow$

$\Rightarrow z(\xi, n) = \log \xi + z(0, n) \Rightarrow z(x,y) = \log x + z(0, x\log x - y)$

Αλλά, $z(1,y) = y \Rightarrow z(1,y) = \log 1 + z(0, 1\cdot\log 1 - y) \Rightarrow$

$\Rightarrow y = z(0, -y) \Rightarrow z(0, -y) = y \Rightarrow$

$\Rightarrow z(0, -(-x\log x + y)) = -x\log x + y \Rightarrow$

$\Rightarrow z(0, x\log x - y) = -x\log x + y$

Άρα, $z(x,y) = \log x - x\log x + y = \log x \cancel{- x + y} = \log x$

Και επαληθεύει την αρχική Δ.Ε